Maths+Music
Video 3:
The Golden Ratio Sequence

Lesson objectives:

Appreciate that a sequence generated with a number that cannot be written as a whole-number fraction (irrational) *never* repeats. Translate sequences into rhythmic drumming patterns and perform them. Understand that our chosen aperiodic sequence can be created using an 'adding rule', and explore how the golden ratio can be found within the growing sequence. See examples of self-similar fractals (including one generated from our sequence), and investigate the fractal properties of the aperiodic sequence. Perform fractal rhythms solo or in a group.

Keywords:

- Irrational Number
- The Golden Ratio (1.61803…)
- Paradiddle
- Self-Similar Fractal

Part 1:

The periodic, deterministic sequences from the last video were created by following paths over a grid of squares that started at a crossroad point and passed through other crossroad points along their journeys. These paths could be described using fractions: the number of squares counted up, divided by the number of squares counted along to reach a crossroads.

These fractions can be seen as the point that the path crosses the first vertical line. Crossing points that can be described as a fraction create periodic sequences.

Irrational numbers cannot be written as a fraction, with one whole number divided by another, and so a path created using an irrational number misses all crossroads and is aperiodic.

Case Study: The Golden Ratio

- 1.61803398875…
    - Numbers after the decimal point form an aperiodic sequence.
    - Cannot be written exactly as a decimal or a fraction

Sequence created from The Golden Ratio:

- ABAABABAABAABABAABABAABAABABAABAABABAABABAABAABA...

Exercise 1:

Practice performing our aperiodic sequence as if it were a musical melody or a drumming pattern.

Choose two sounds to be A and B, or play each of them with either your left or right hand.

To make it easier, start slowly and split the sequence into chunks of 8:

ABAABABA
ABAABABA
ABABAABA
ABABAABA
ABABAABA
BAABAABA

Note: The first 2 lines are the same (ABAABABA) as are lines 3-5 (ABABAABA)

Part 2:

As travelling along a straight path is a process that includes no randomness, it is a deterministic system. We know from the last video that deterministic systems have rules, so is there a rule that we can use to build this sequence?

- Start with just A and B. Think of A as the 'first part' and B the 'second part'.
- To grow the sequence you add the first part on to the end of the bit of sequence you already have:
  - AB + A = ABA
- You can think of your new sequence as being made from the section of sequence you had before, AB (the first part), and the section you added on, A (the second part).
- Add the 'first part' on the end of your sequence again:
  - ABA + AB = ABAAB
- You can continue to do this to create the sequence step by step:
  - ABAAB + ABA = ABAABABA
  - ABAABABA + ABAAB = ABAABABAABAAB
  - ABAABABAABAAB + ABAABABA = ABAABABAABAABABAABABA

- (Side note: this adding rule has the same mathematical structure as the rule for generating the 'Fibonacci sequence' 1, 1, 2, 3, 5, 8, 13, 21, …, where each term is the sum of the previous two.)

Exercise 2:

Build the sequence in steps, by taking the previous section of the sequence, and adding on the section before that.

A
AB
ABA
ABAAB
ABAABABA
ABAABABAABAAB

Suggestion: have pupils build the sequence incrementally using blocks (pick different colours to represent A & B). Alternatively, write the sequences down in exercise books and see who can get the furthest.

Extension: in groups of 3 or more, play the market game as described in the video.

Bonus Exercise:

This question is raised in the section of the video following Exercise 2:

If you divide the number of As by the number of Bs in these increasingly long sections of the sequence, you get these numbers:

- ABAABABAABAAB: 8 / 5 = 1.6
- ABAABABAABAABABAABABA: 13 / 8 = 1.625
- ABAABABAABAABABAABABAABAABABAABAAB: 21 / 13 = 1.61538…

If you took a really long section of the sequence and divided the number of As by the number of Bs, what number would you get?

ANSWER: As the section of the sequence gets longer, the number of As divided by the number of Bs gets closer and closer to the Golden Ratio (1.61803398875…)

Part 3A (visual fractals):

Self-similar fractals are objects that contain smaller copies of themselves if you zoom in and look at them more closely.

A snowflake is an example of a self-similar fractal occurring in nature: each of its arms has smaller arms, each a copy of the original, branching off them, and those arms have even smaller arms branching off them.

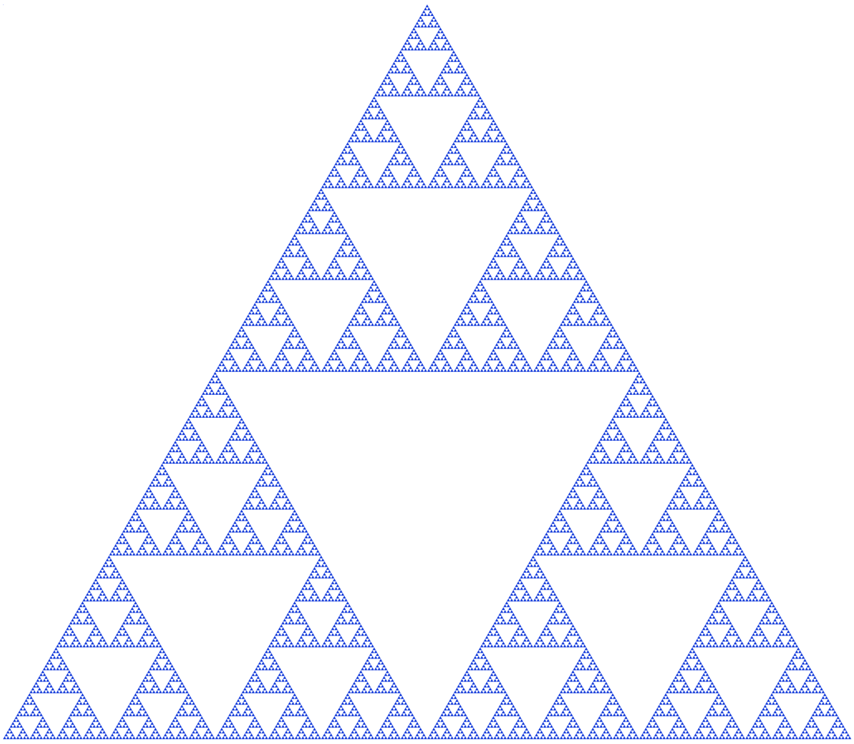
Musical Reference: *Let it Go,* from 'Frozen'
https://www.youtube.com/watch?v=L0MK7qz13bU

Other examples of fractals (not shown in the video):

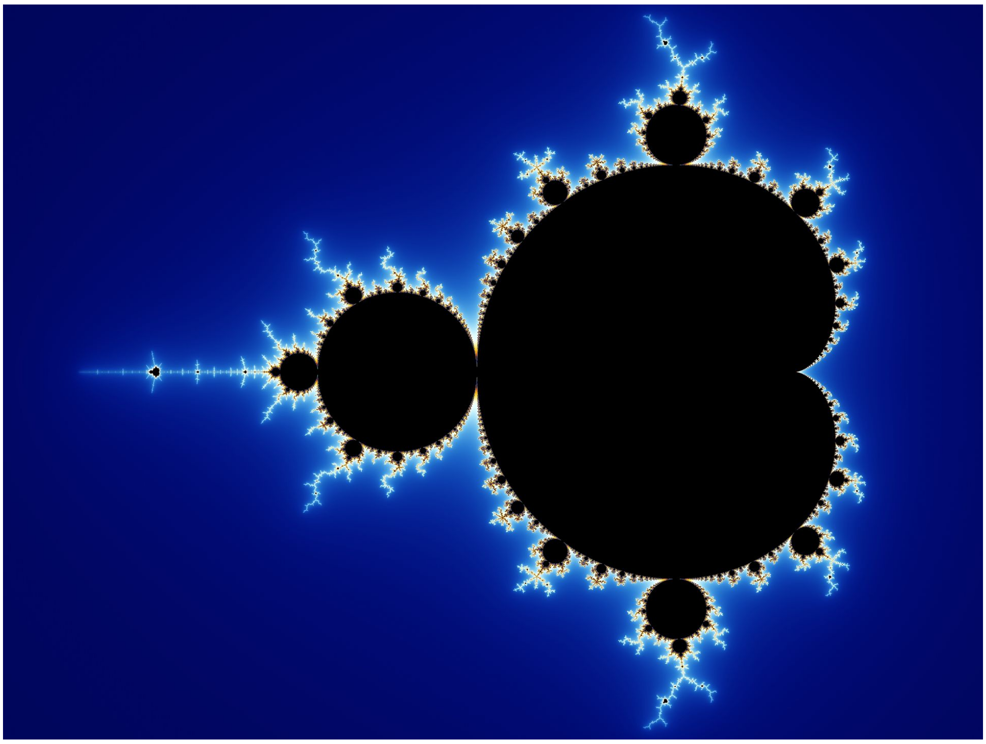
There are many other examples in nature - e.g. ferns (each sub-branch is a smaller copy of the plant),

romanescu broccoli (each bud is a smaller copy of the whole and contains smaller and smaller buds)

And mathematically generated fractals include the Sierpinski triangle



And the famous Mandelbrot set, named after the pioneer of fractal geometry, Benoit Mandelbrot



Our aperiodic sequence can create a self-similar fractal in the following way:

- Replace every A with a kick drum and every B with a snare drum, and listen to the pattern whilst walking with one step to each beat.
- Every time you hear a kick drum (and A), turn in the direction of the foot that has just been placed down
  - Left foot = turn 90 degrees left
  - Right foot = turn 90 degrees right

Watching the path that this sequence creates, see how large sections of the path have smaller sections within them that outline the same shape, and that those smaller sections outline the same shapes within them. The name of this fractal is the 'Fibonacci word fractal' (https://en.wikipedia.org/wiki/Fibonacci_word_fractal).

NOTE: There is a separate video showing the full fractal path, accompanied by music, that might work well as a background video as the students work (https://youtu.be/PG_PMYZ7BPw).

Part 3B (musical fractals):

To explore the same fractal properties found within our sequence musically, we can look again at how it was created:

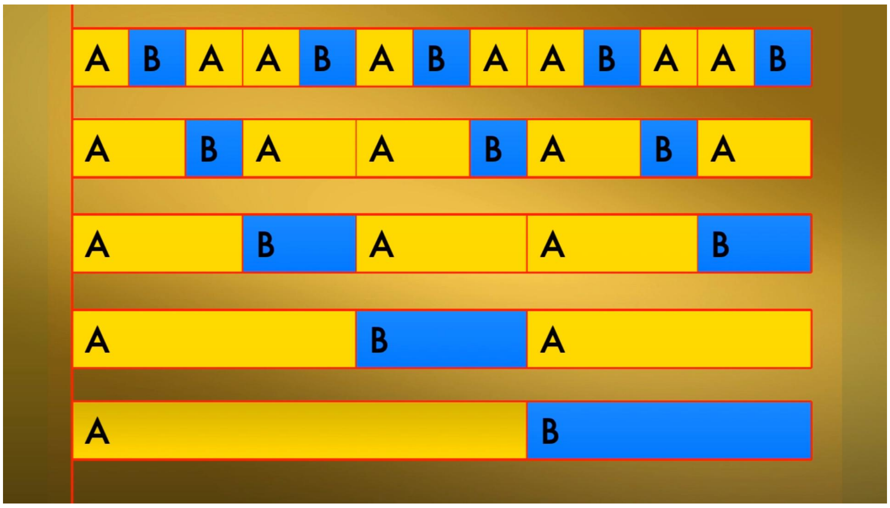
- We began with a chunk of AB, and then added on an A.
- Everything after this involved repetitions of what had already happened.
- Because of this, we can break down our sequence into chunks of AB, and A.
- Looking at the order the chunks of AB and A appear in, we find a copy of the original sequence: ABAABABA...

Each new version of the sequence found this way has As and Bs of different lengths. The sequence has been 'stretched out'.

- 1st new sequence has As length 2, Bs length 1
- 2nd new sequence has As length 3, Bs length 2
- 3rd new sequence has As length 5, Bs length 3
- Eventually, As will be one Golden Ratio times longer than the Bs

Exercise 3:

Practice playing the stretched out versions of the sequence alongside the original drumming pattern. The length of As and Bs changes with each new version of the sequence.



NOTE: There are separate videos for each new added layer. You can change the speed of the youtube video to make it slower or faster as necessary. (https://www.youtube.com/playlist?list=PLSbPVkZDes0TzaNzmBP1_3w_3p8r4ljl0)

We suggest playing along as a whole class, using chime bars, drums or tapping the table. Start slow and easy - e.g. one or two groups, and then progress to three or more groups if pupils are confident

It might be useful to relate this exercise to the shape-tapping exercise in video 2, as a way of helping the students count the length of As and Bs:

- Layer 1: ABAABABAABAAB
    - A = 1 (single point)
    - B = 1 (single point)

- Layer 2: ABAABABA
    - A = 2 (line / two points joined)
    - B = 1 (single point)
    - Sequence: 2 - 1 - 2 - 2 - 1 - 2 - 1 - 2

- Layer 3: ABAAB
    - A = 3 (triangle)
    - B = 2 (line two points joined)
    - Sequence: 3 - 2 - 3 - 3 - 2

- Layer 4: ABA
    - A = 5 (pentagon)
    - B = 3 (triangle)
    - Sequence: 5 - 3 - 5

- Layer 5: AB
  - A = 8 (octagon)
  - B = 5 (pentagon)
  - Sequence = 8 - 5